Ranking of fuzzy numbers based on angle measure

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ABSTRACT
In this paper, a novel approach for ranking fuzzy numbers based on the angle measure is introduced. Several left and right spreads at each chosen \( \alpha \)-levels of fuzzy numbers is used to determine center of mass points (CMPs) and then, the angles between the CMPs and the horizontal axis is calculated. The total angle is determined by averaging the computed angles and finally, the novel method is compared with other methods by solving some numerical examples.

Key words: Ranking of fuzzy numbers; center of mass; Angle measure

1. Introduction
In many decision making procedures such as [19, 28], two or more than two quantity must be compared and hence, in fuzzy environment, ranking of fuzzy numbers is a very important decision making procedures. In the first proposed method, Jain [20, 21] employed the concept of maximizing set to order the fuzzy numbers in 1976 and 1978 and after his work, many authors have investigated various ranking methods [1, 3, 4, 5, 7, 10, 11, 12, 13, 14, 16, 17, 18, 22, 24, 25, 26, 27, 31, 32, 33]. Some of these ranking methods have been compared and reviewed by Bortolan and Degani [8], and more recently by Chen and Hwang [9]. Some of important and applicable contributions in this field include: an index for ordering fuzzy numbers defined by Choobineh and Li [10], ranking alternatives using fuzzy numbers studied by Dias [16], automatic ranking of fuzzy numbers using artificial neural networks proposed by Requena et al. [27], ranking fuzzy values with satisfaction function investigated by Lee et al. [22]. Ranking and defuzzification methods based on area compensation presented by Fortemps and Roubens [18], and ranking alternatives with fuzzy weights using maximizing set and minimizing set given by Raj and Kumar [25]. Abbasbandy and Asady give method to rank the fuzzy numbers by sign distance [1]. However, some of these methods are computationally complex and difficult to implement, and others are counterintuitive and not discriminating. Furthermore, many of them produce different ranking outcomes for the same problem [12].

In spite of many ranking methods, no one can rank fuzzy numbers with human intuition consistently in all cases, in this work, we propose a simple ranking method for triangular (trapezoidal) numbers, which associates an intuitive geometrical representation (near to the center of gravity). The proposed ranking method has basic mathematical properties. It does not imply much computational effort and does not require a priori knowledge of the set of all alternatives.

The rest of Paper is organized as follows. Section 2, contains the basic definitions and notations used in the remaining parts of the paper. In Section 3, we introduce the ranking approach based angle measure and describes some useful properties. In Section 4, solving some examples and compare angle measure with other methods. Concluding remarks are finally made in section 5.

2. Basic definitions and notation
A real fuzzy number can be defined as a fuzzy subset of the real line \( R \), which is convex and normal. That is, for a fuzzy number \( A \) of \( R \) defined by the membership function \( \mu_A(x) \), \( x \in R \). The following relations exist:
\[
\max_a \mu_A(x) = 1
\]
\[
\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\},
\]
Where \( x_1, x_2 \in R, \forall \lambda \in [0,1] \). A fuzzy number \( A \) with the membership function \( \mu_A(x), x \in R \) the fuzzy number \( A = [a, b, c, d; 1] \) can be defined as
\[ \mu_A(x) = \begin{cases} \mu_A^L(x) & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \mu_A^R(x) & c \leq x \leq d \\ 0 & \text{otherwise.} \end{cases} \]

Where \( \mu_A^L(x) \) is the left membership function that is an increasing function and \( \mu_A^L : [a, b] \rightarrow [0, 1] \). Meanwhile, \( \mu_A^R(x) \) is the right membership function that is a decreasing function and \( \mu_A^R : [c, d] \rightarrow [0, 1] \). For a fuzzy number \( A_i \), the \( \alpha \)-cuts (level sets) \( A_x = \{ x \in \mathbb{R} | \mu_A(x) \geq \alpha \} \), \( \alpha \in [0, 1] \), are convex subsets of \( \mathbb{R} \). The lower and upper limits of the \( k^{th} \) -cut for the fuzzy number \( A_i \) are defined as

\[
\begin{align*}
l_{ik} &= \inf \{ x \in \mathbb{R} | \mu_A(x) \geq \alpha_k \} \\
r_{ik} &= \sup \{ x \in \mathbb{R} | \mu_A(x) \geq \alpha_k \}
\end{align*}
\]

respectively, where \( l_{ik} \) and \( r_{ik} \) are left and right spreads, and \( \alpha_k = \frac{k}{n} \), for \( k = 0, \ldots, n \) [13].

3. Ranking of fuzzy numbers by the angle measure

In this Section the angle measure method (AMM) is described. AMM rank fuzzy numbers based on the average of angles between horizontal axis and the line that joins the origin and CMPs of each fuzzy number. Let \( A_i \), \( i = 0, \ldots, I \) are fuzzy numbers, to define the angle measure of \( A_i \), the following steps should be done:

- Left and right spreads at \( k \) \( \alpha \)-levels of fuzzy numbers, \( l_{ik}, r_{ik} \) is calculated.
- CMPs of each fuzzy numbers \( A_i \) which denoted by \( (x_{ik}, y_{ik}) \) is determined as below:

\[
\overline{m_{\alpha_k}} = \int_{l_{ik}}^{r_{ik}} x \alpha_k dx = \frac{1}{2} \alpha_k (r_{ik}^2 - l_{ik}^2)
\]

\[
m_{\alpha_k} = \int_{l_{ik}}^{r_{ik}} \alpha_k dx = \alpha_k (r_{ik} - l_{ik})
\]

\[
x_{ik} = \frac{m_{\alpha_k}}{m_{\alpha_k}}
\]

\[
y_{ik} = \alpha_k
\]

- \( \theta_{ik} \) of the \( k^{th} \) -cuts for the fuzzy number \( A_i \) is determined as

\[
\theta_{ik} = \arctan \left( \frac{y_{ik}}{x_{ik}} \right)
\]

• Total angle is calculated as following

\[
\theta_i = \frac{\sum_{k=1}^{n} \theta_{ik}}{n + 1}
\]

\[
\theta_i = \begin{cases} 
\theta_i, & \text{if } \theta_i > 0 \\
\theta_i + 180, & \text{if } \theta_i < 0
\end{cases}
\]

Figure 1. Angles for some different \( \alpha \)-levels of fuzzy numbers. An arbitrary \((x_{ik}, y_{ik})\) is determined.

3.1. Properties

We consider the following reasonable properties for the ordering approaches, see [30].

\( A_i \): For an arbitrary finite subset \( \Gamma \) of \( E \) and \( A \in \Gamma \), \( A \pm A \).

\( A_2 \): For an arbitrary finite subset \( \Gamma \) of \( E \) and \( (A, B) \in \Gamma^2 \), \( A \pm B \) and \( B \pm A \), we should have \( A \approx B \).
$A_1$: For an arbitrary finite subset $\Gamma$ of $E$ and $(A, B, C) \in \Gamma^3$, $A \pm B$ and $B \pm C$, we should have $A \pm C$.

$A_2$: For an arbitrary finite subset $\Gamma$ of $E$ and $(A, B) \in \Gamma^2$, $\text{inf} \ supp(A) \geq \text{sup} \ supp(B)$, we should have $A \pm B$.

$A_3$: For an arbitrary finite subset $\Gamma$ of $E$ and $(A, B) \in \Gamma^2$, $\text{inf} \ supp(A) > \text{sup} \ supp(B)$, we should have $A \succ B$.

$A_4$: Let $\Gamma$ and $\Gamma'$ be two arbitrary finite subsets of $E$ in which $A$ and $B$ are in $\Gamma \cap \Gamma'$. We obtain the ranking order $A \succ B$ by (1) on $\Gamma$ if and only if $A \succ B$ by (1) on $\Gamma'$.

Remark 3.1 A novel method based on Definition 3.1 has the properties $A_1, A_2, A_3, A_4, A_3', A_3$. Proof: It is easy to verify that the properties $A_1, A_2, A_3, A_4, A_3, A_3'$ are hold.

Remark 3.2 The simple form of ranking formula for triangular fuzzy number is as follow:

$$x_{ik} = \frac{1}{2}(a + c)(1 - \alpha_{ik}) + b\alpha_{ik} \quad \text{and} \quad \gamma_{ik} = \alpha_{ik}$$

$$\theta_{ik} = \arctan\left(\frac{1}{2}(a + c)(\frac{1}{\alpha_{ik}} - 1) + b\right)$$

4. Numerical Examples

Example 4.1.
Consider the following sets, see Yao and Wu [31].
Set 1: $A=(0.4, 0.5, 1)$, $B=(0.4, 0.7, 1)$, $C=(0.4, 0.9, 1)$.
Set 2: $A=(0.3, 0.4, 0.7, 0.9)$, $B=(0.3, 0.7, 0.9)$, $C=(0.5, 0.7, 0.9)$.
Set 3: $A=(0.3, 0.5, 0.7)$, $B=(0.3, 0.5, 0.8, 0.9)$, $C=(0.3, 0.5, 0.9)$.
Set 4: $A=(0.0, 0.4, 0.7, 0.8)$, $B=(0.2, 0.5, 0.9)$, $C=(0.1, 0.6, 0.8)$.
Table 1. Comparative results of Example 4.1

<table>
<thead>
<tr>
<th>Authors</th>
<th>Fuzzy number</th>
<th>set 1</th>
<th>set 2</th>
<th>set 3</th>
<th>set 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A angle measure method for n=10</td>
<td>A</td>
<td>36.3347°</td>
<td>36.6467°</td>
<td>39.5242°</td>
<td>40.2134°</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>32.1914°</td>
<td>33.4474°</td>
<td>34.4644°</td>
<td>38.6373°</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>28.7557°</td>
<td>32.1914°</td>
<td>37.8154°</td>
<td>37.9847°</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Cheng and Li</td>
<td>A</td>
<td>0.333</td>
<td>0.458</td>
<td>0.333</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.50</td>
<td>0.583</td>
<td>0.467</td>
<td>0.5833</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.667</td>
<td>0.667</td>
<td>0.5417</td>
<td>0.6111</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Yager</td>
<td>A</td>
<td>0.60</td>
<td>0.575</td>
<td>0.5</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.70</td>
<td>0.65</td>
<td>0.55</td>
<td>0.525</td>
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<tr>
<td></td>
<td>C</td>
<td>0.80</td>
<td>0.7</td>
<td>0.625</td>
<td>0.55</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Chen</td>
<td>A</td>
<td>0.3375</td>
<td>0.4315</td>
<td>0.375</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.50</td>
<td>0.5625</td>
<td>0.425</td>
<td>0.57</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.667</td>
<td>0.625</td>
<td>0.55</td>
<td>0.625</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Baldwin and Guild</td>
<td>A</td>
<td>0.30</td>
<td>0.27</td>
<td>0.27</td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.33</td>
<td>0.27</td>
<td>0.37</td>
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<tr>
<td></td>
<td>C</td>
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<td>0.37</td>
<td>0.37</td>
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<tr>
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<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Chu and Taso</td>
<td>A</td>
<td>0.299</td>
<td>0.2847</td>
<td>0.25</td>
<td>0.24402</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.350</td>
<td>0.32478</td>
<td>0.31526</td>
<td>0.26243</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.3993</td>
<td>0.350</td>
<td>0.27475</td>
<td>0.2619</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ C $\prec$ B</td>
<td></td>
</tr>
<tr>
<td>Yao and Wu</td>
<td>A</td>
<td>0.6</td>
<td>0.575</td>
<td>0.5</td>
<td>0.475</td>
</tr>
<tr>
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<td>B</td>
<td>0.7</td>
<td>0.65</td>
<td>0.625</td>
<td>0.525</td>
</tr>
<tr>
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<td>C</td>
<td>0.8</td>
<td>0.7</td>
<td>0.55</td>
<td>0.525</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Sign Distance Method p=1</td>
<td>A</td>
<td>1.2</td>
<td>1.15</td>
<td>1</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.4</td>
<td>1.3</td>
<td>1.25</td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>1.6</td>
<td>1.4</td>
<td>1.1</td>
<td>1.05</td>
</tr>
<tr>
<td>Results</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Sign Distance Method p=1</td>
<td>A</td>
<td>0.8869</td>
<td>0.8756</td>
<td>0.7257</td>
<td>0.7853</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.0194</td>
<td>0.9522</td>
<td>0.9416</td>
<td>0.7958</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>1.1605</td>
<td>1.0033</td>
<td>0.8165</td>
<td>0.8386</td>
</tr>
<tr>
<td>Results</td>
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<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ B $\prec$ C</td>
<td></td>
</tr>
<tr>
<td>Cheng Distance</td>
<td>A</td>
<td>0.79</td>
<td>0.7577</td>
<td>0.7071</td>
<td>0.7106</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.8602</td>
<td>0.8149</td>
<td>0.8037</td>
<td>0.7256</td>
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<tr>
<td></td>
<td>C</td>
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<tr>
<td>Results</td>
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<td>A $\prec$ B $\prec$ C</td>
<td>A $\prec$ C $\prec$ B</td>
<td>A $\prec$ C $\prec$ B</td>
<td></td>
</tr>
<tr>
<td>Cheng CV uniform distribution</td>
<td>A</td>
<td>0.0272</td>
<td>0.0328</td>
<td>0.0133</td>
<td>0.0693</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0.0214</td>
<td>0.0246</td>
<td>0.0304</td>
<td>0.0385</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.0225</td>
<td>0.0095</td>
<td>0.0275</td>
<td>0.0433</td>
</tr>
<tr>
<td>Results</td>
<td>B $\prec$ C $\prec$ A</td>
<td>C $\prec$ B $\prec$ A</td>
<td>A $\prec$ C $\prec$ B</td>
<td>B $\prec$ C $\prec$ A</td>
<td></td>
</tr>
<tr>
<td>Cheng CV proportional distribution</td>
<td>A</td>
<td>0.0183</td>
<td>0.026</td>
<td>0.008</td>
<td>0.0471</td>
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<tr>
<td></td>
<td>B</td>
<td>0.0128</td>
<td>0.0146</td>
<td>0.0234</td>
<td>0.0236</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0.0137</td>
<td>0.0057</td>
<td>0.0173</td>
<td>0.0255</td>
</tr>
<tr>
<td>Results</td>
<td>B $\prec$ C $\prec$ A</td>
<td>C $\prec$ B $\prec$ A</td>
<td>A $\prec$ C $\prec$ B</td>
<td>B $\prec$ C $\prec$ A</td>
<td></td>
</tr>
</tbody>
</table>

Example 4.2 Consider the following sets, see L-H Chen and H-W Lu [13].

Set 1: $A_1 = (0.3,0.5,0.7), A_2 = (0.1,0.2,0.3)$.
Set 2: $E_1 = (0.2,0.5,0.8), E_2 = (0.3,0.4,0.9)$.
Set 3: $B_1 = (0.7,0.8,0.9), B_2 = (0.0,0.4,1)$.
Set 4: $F_1 = (0.2,0.5,0.8), F_2 = (0.3,0.3,0.9)$.

Set 5: $C_1 = (0.4,0.7,1), C_2 = (0.1,0.4,0.9)$.
Set 6: $I_1 = (0.4,0.6,0.8), I_2 = (0.3,0.6,0.8)$.
Set 7: $G_1 = (0.3,0.5,0.7), G_2 = (0.1,0.5,0.9)$.
Set 8: $D_1 = (0.0,0.5,1), D_2 = (0.3,0.4,1)$.
Set 9: $J_1 = (0.4,0.6,0.8), J_2 = (0.3,0.4,0.7,0.8)$.
In Table 2 all previous methods give $E_1 < E_2$, which is an intuition contradiction but AMM gives $E_2 < E_1$.

5. Conclusions
In this paper, we have presented a simple ranking method for triangular (trapezoidal) fuzzy numbers, which associates an intuitive geometrical representation. It does not imply much computational effort and does not require a priori knowledge of the set of all alternatives. Comparative examples illustrate the advantage of the new approach.

References